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Branes with fluxes wrapped on spheres

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Abstract: Following an eight-dimensional gauged supergravity approach we construct the most general solution describing D6-branes wrapped on a Kähler four-cycle taken to be the product of two spheres of different radii. Our solution interpolates between a Calabi-Yau four-fold and the spaces $S^2 \times S^2 \times S^2 \times \mathbb{R}^2$ or $S^2 \times S^2 \times \mathbb{R}^4$, depending on generic choices for the parameters. Then we turn on a background four-form field strength, corresponding to D2-branes, and show explicitly how our solution is deformed. For a particular choice of parameters it represents a flow from a Calabi-Yau four-fold times the three-dimensional Minkowski space-time in the ultraviolet, to the space-time $AdS_4 \times Q^{1,1,1}$ in the infrared. In general, the solution in the infrared has a singularity which within type-IIA supergravity corresponds to the near horizon geometry of the solution for the D2-D6 system. Finally, we uncover the relation with work done in the eighties on Freund-Rubin type compactifications.

Keywords: M-Theory, D-branes, Superstring Vacua, AdS-CFT and dS-CFT Correspondence.

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Contents

Branes wrapped on supersymmetric cycles provide a natural path to obtain gravity duals of field theories with low supersymmetry. These field theories are twisted since preserving some supersymmetry after wrapping the brane, requires the identification (expressed better, the relation) of the spin connection on the cycle and some external R-symmetry gauge fields [1]. Therefore, the dual supergravity solutions can be naturally constructed in an appropriate gauged supergravity, and are eventually lifted to ten or eleven dimensions. This approach was started in [2], and has been further developed for a wide variety of branes wrapped on diverse supersymmetric cycles [3]–[21].

The case of D6-branes is of special interest because they lift to pure geometry in eleven dimensions. This fact allows to argue how compactifications of M-theory on manifolds with reduced holonomy arise as the local eleven dimensional description of D6-branes wrapped on supersymmetric cycles in manifolds of lower dimension with a different holonomy group [22]. This extends the work of [23], where D6-branes wrapping the three-cycle in the deformed conifold were shown to be described in eleven dimensions as a compactification on a seven manifold with G_2 holonomy. These lifts to eleven dimensions for D6-branes wrapping various cycles were explicitly constructed using eight dimensional gauged supergravity [24] in [7, 11, 13, 19].

However these purely gravitational geometries are deformed when background fluxes are included. In [13] M-theory on a Calabi-Yau four-fold was shown to arise as the eleven dimensional description of D6-branes wrapped on Kähler four-cycles inside Calabi-Yau three-folds. The deformation of this background by a four-form field strength along the unwrapped coordinates was recently considered in [21], where it was shown to induce a flow from $\mathbb{E}_{2,1} \times CY_4$ at ultraviolet to $AdS_4 \times Q^{1,1,1}$ in the infrared limit.

The four-cycle in [13, 21] was taken to be a product of two two-spheres of the same radius so that the metric was Einstein. In this paper we will eliminate the Einstein condition on the four-cycle allowing the spheres to have different radii and will also introduce a four-form flux. When lifted to eleven dimensions, and in the absence of flux, our solution will represent M-theory on a Calabi-Yau four-fold. We will find a three parameter family of metrics in which the conical singularity of the four-fold is generically resolved by being replaced by a *bolt* or *nut* singularity which is removable [25]. Then we turn on a background

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four-form field strength, corresponding to D2-branes, which provides another mechanism for resolving the singularity. After determining the most general supersymmetry preserving solution we discuss its behavior for various choices for the parameters. A special choice of parameters leads to an eleven-dimensional solution that flows from a Calabi-Yau four-fold times the three-dimensional Minkowski space-time in the ultraviolet, to the space-time $AdS_4 \times Q^{1,1,1}$ in the infrared, where $Q^{1,1,1}$ is the seven-manifold coset space $\mathrm{SU}(2)^3/\,\mathrm{U}(1)^2$ that is supersymmetric [26]. While this is similar to [21], in the general case the singularity persists and is the same as in the near horizon metric for the D2-D6 system. Finally, we end the paper by making a precise connection of our work with Freund-Rubin type compactifications of eleven-dimensional supergravity to four dimensions.

Before constructing our solution we will briefly review some relevant facts about gauged supergravity in eight dimensions which was constructed by Salam and Sezgin [24] through Scherk-Schwarz compactification of eleven-dimensional supergravity [27] on an SU(2) group manifold. The field content of the theory consists of the metric $g_{\mu\nu}$, a dilaton Φ, five scalars given by a unimodular 3×3 matrix L^i_α in the coset $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ and an SU(2) gauge potential A_μ, all in the gravity sector, and a three-form coming from reduction of the eleven dimensional three-form.[1] In addition, on the fermion side we have the pseudo-Majorana spinors ψ_μ and χ_i.

The lagrangian density for the bosonic fields is given, in $\kappa = 1$ units, by

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{4}R - \frac{1}{4}e^{2\Phi}F^{\alpha}_{\mu\nu}F^{\mu\nu\,\beta}g_{\alpha\beta} - \frac{1}{4}P_{\mu\,ij}P^{\mu\,ij} - \frac{1}{2}(\partial_\mu\Phi)^2 - \\ & - \frac{g^2}{16}e^{-2\Phi}\left(T_{ij}T^{ij} - \frac{1}{2}T^2\right) - \frac{1}{48}e^{2\Phi}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\,,\end{aligned} \tag{1}$$

where $F^{\alpha}_{\mu\nu}$ is the Yang-Mills field strength. Supersymmetry will be preserved by bosonic solutions to the equations of motion of eight dimensional supergravity if the supersymmetry variations for the gaugino and the gravitino vanish. These are, respectively, given by

$$\begin{aligned}\delta\chi_i = {}& \frac{1}{2}(P_{\mu\,ij} + \frac{2}{3}\delta_{ij}\partial_\mu\Phi)\hat{\Gamma}^j\Gamma^\mu\epsilon - \frac{1}{4}e^{\Phi}F_{\mu\nu\,i}\Gamma^{\mu\nu}\epsilon - \\ & - \frac{g}{8}e^{-\Phi}\left(T_{ij} - \frac{1}{2}\delta_{ij}T\right)\epsilon^{jkl}\hat{\Gamma}_{kl}\epsilon - \frac{1}{144}e^{\Phi}G_{\mu\nu\rho\sigma}\hat{\Gamma}_i\Gamma^{\mu\nu\rho\sigma}\epsilon = 0\,,\end{aligned} \tag{2}$$

and

$$\begin{aligned}\delta\psi_\gamma = {}& \mathcal{D}_\gamma\epsilon + \frac{1}{24}e^{\Phi}F^i_{\mu\nu}\hat{\Gamma}_i(\Gamma_\gamma^{\mu\nu} - 10\delta_\gamma^\mu\Gamma^\nu)\epsilon - \\ & - \frac{g}{288}e^{-\Phi}\epsilon_{ijk}\hat{\Gamma}^{ijk}\Gamma_\gamma T\epsilon - \frac{1}{96}e^{\Phi}G_{\mu\nu\rho\sigma}(\Gamma_\lambda^{\mu\nu\rho\sigma} - 4\delta_\lambda^\mu\Gamma^{\nu\rho\sigma})\epsilon = 0\,,\end{aligned} \tag{3}$$

where the covariant derivative is

$$\mathcal{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon + \frac{1}{4}Q_{\mu\,ij}\hat{\Gamma}^{ij}\epsilon\,, \tag{4}$$

[1] Reduction of the eleven-dimensional three-form also produces a scalar, three vector fields and three two-forms. However, we will set all these fields to zero.

while $P_{\mu\,ij}$ and $Q_{\mu\,ij}$ are, respectively, the symmetric and antisymmetric quantities entering the Cartan decomposition of the $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ coset, defined through

$$P_{\mu\,ij} + Q_{\mu\,ij} \equiv L_i^\alpha(\partial_\mu \delta_\alpha^\beta - g\,\epsilon_{\alpha\beta\gamma}A_\mu^\gamma)L_{\beta\,j}\,, \tag{5}$$

and T_{ij} is the T-tensor defining the potential energy associated to the scalar fields,

$$T^{ij} \equiv L_\alpha^i L_\beta^j \delta^{\alpha\beta}\,, \tag{6}$$

with $T \equiv T_{ij}\delta^{ij}$, and

$$L_\alpha^i L_j^\alpha = \delta_j^i, L_\alpha^i L_\beta^j \delta_{ij} = g_{\alpha\beta}, L_\alpha^i L_\beta^j g^{\alpha\beta} = \delta^{ij}\,. \tag{7}$$

As usual, curved directions are labeled by greek indices, while flat ones are labeled by latin, and $\mu, a = 0, 1, \ldots, 7$ are space-time coordinates, while $\alpha, i = 8, 9, 10$ are in the group manifold. Note also that upper indices in the gauge field, A_μ^α, are curved, and that the field strength in eight dimensional curved space is defined as

$$G_{\mu\nu\rho\sigma} = e^{-4\Phi/3} e_\mu^a e_\nu^b e_\rho^c e_\sigma^d F_{abcd}\,. \tag{8}$$

The 32×32 gamma matrices in eleven dimensions can be represented as

$$\Gamma^a = \gamma^a \times \mathbb{1}_2, \hat{\Gamma}^i = \gamma_9 \times \tau^i\,, \tag{9}$$

where the γ_a's denote the 16×16 gamma matrices in eight dimensions and as usual $\gamma_9 = i\gamma^0\gamma^1 \ldots \gamma^7$, so that $\gamma_9^2 = \mathbb{1}$, while τ^i are Pauli matrices. It will prove useful to introduce $\Gamma_9 \equiv \frac{1}{6i}\epsilon_{ijk}\hat{\Gamma}^{ijk} = -i\hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3 = \gamma_9 \times \mathbb{1}_2$.

Let us now present the system under study in this paper and construct a solution describing this configuration. We will consider a D2-D6 brane system, with the D6-branes wrapped on a Kähler four-cycle inside a Calabi-Yau three-fold, and the D2-branes along the unwrapped directions. Keeping some supersymmetry unbroken involves an identification of the spin connection of the supersymmetric cycle and the gauge connection of the structure group of the normal bundle. When we wrap the D6-branes on the four-cycle, the $\mathrm{SO}(1,6) \times \mathrm{SO}(3)_R$ symmetry group of the branes in flat space is broken to $\mathrm{SO}(1,2) \times \mathrm{SO}(4) \times \mathrm{U}(1)_R$. The breaking of the R-symmetry takes place because there are two normal directions to the D6-branes that are inside the Calabi-Yau three-fold; the R-symmetry is therefore broken to the $\mathrm{U}(1)_R$ corresponding to them. The twisting will amount to the identification of this $\mathrm{U}(1)_R$ with a $\mathrm{U}(1)$ subgroup in one of the $\mathrm{SU}(2)$ factors in $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$. The remaining scalar after the twisting, together with the vector and two fermions preserved by the diagonal group of $\mathrm{U}(1) \times \mathrm{U}(1)_R$, determine the field content of $\mathcal{N} = 2$ three-dimensional Yang-Mills. In the absence of D2-branes the lift to eleven-dimensions corresponds to M-theory on a Calabi-Yau four-fold [22, 13].

We will choose the four-cycle to be a product of two two-spheres of different radii, $S^2 \times \bar{S}^2$. The deformation on the world-volume of the D6-branes will then be described by a metric of the form

$$ds_8^2 = e^{2f} ds_{1,2}^2 + d\rho^2 + \alpha^2 d\Omega_2^2 + \beta^2 d\bar{\Omega}_2^2 \,, \tag{10}$$

where $ds_{1,2}^2$ is the three-dimensional Minkowski metric, the line elements for the two-spheres are

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2 \,, \qquad d\bar{\Omega}_2^2 = d\bar{\theta}^2 + \sin^2\bar{\theta} d\bar{\phi}^2 \,, \tag{11}$$

and f, α and β depend only on the radial variable ρ. The same will be true for all additional fields that we will turn on. The four-form flux implied by the D2-branes along the unwrapped directions will be

$$G_{x_0 x_1 x_2 \rho} = Q \frac{e^{-2\Phi+3f}}{\alpha^2\beta^2} \,, \tag{12}$$

where in the above x_0, x_1, x_2 are curved directions, Q is a dimensionfull constant and the specific functional dependence is uniquely fixed by the equation of motion for the three-form potential.

Turning now to the Killing spinor equation we should observe that it is quite useful to introduce the triplet of Maurer-Cartan 1-forms on S^2

$$\sigma_1 = \sin\theta d\phi \,, \qquad \sigma_2 = d\theta \,, \qquad \sigma_3 = \cos\theta d\phi \,. \tag{13}$$

We note that they obey the conditions $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_j$, so that they resemble the triplet of Maurer-Cartan forms on S^3, but obviously only two of them are the independent ones. We also introduce a similar triplet $\bar{\sigma}_i$ defined on the other sphere $\bar{S}^2$.

Consistency of the Killing spinor equations after splitting of the four-cycle into the product of spheres in (10) requires turning on only one of the components of the gauge field,

$$A^3 = -\frac{1}{g}(\sigma_3 + \bar{\sigma}_3) \,, \tag{14}$$

thus realizing the breaking of the $\mathrm{SU}(2)_R$ R-symmetry to $\mathrm{U}(1)_R$ required by the twisting. In addition, consistency requires turning on one of the scalars in L^i_α,

$$L^i_\alpha = \mathrm{diag}\,(e^\lambda, e^\lambda, e^{-2\lambda}) \,, \tag{15}$$

and imposing on the spinor the projections

$$\begin{aligned} \Gamma_7\epsilon &= -i\Gamma_9\epsilon \,, \\ \Gamma_1\Gamma_2\epsilon &= \bar{\Gamma}_1\bar{\Gamma}_2\epsilon = -\hat{\Gamma}_1\hat{\Gamma}_2\epsilon \,. \end{aligned} \tag{16}$$

These leave in total four independent components for the spinor. We note here that the simple relation (14), stating the equality (up to a constant) of the gauge field and the spin connection, is not valid in general when non-trivial scalar fields are present and instead it gets modified [19].

With the above ansatz and projections on the spinor, the supersymmetry variations (2) and (3) give the following conditions

$$
\begin{aligned}
\frac{d\Phi}{d\rho} &= \frac{g}{8}e^{-\Phi}(e^{-4\lambda}+2e^{2\lambda}) - \frac{1}{2g}e^{\Phi-2\lambda}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right) - \frac{Q}{2}\frac{e^{-\Phi}}{\alpha^2\beta^2}\,,\\
\frac{d\lambda}{d\rho} &= \frac{g}{6}e^{-\Phi}(e^{-4\lambda}-e^{2\lambda}) + \frac{1}{3g}e^{\Phi-2\lambda}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right),\\
\frac{1}{\alpha}\frac{d\alpha}{d\rho} &= \frac{g}{24}e^{-\Phi}(2e^{2\lambda}+e^{-4\lambda}) + \frac{1}{6g}e^{\Phi-2\lambda}\left(\frac{5}{\alpha^2}-\frac{1}{\beta^2}\right) - \frac{Q}{2}\frac{e^{-\Phi}}{\alpha^2\beta^2}\,,\\
\frac{1}{\beta}\frac{d\beta}{d\rho} &= \frac{g}{24}e^{-\Phi}(2e^{2\lambda}+e^{-4\lambda}) + \frac{1}{6g}e^{\Phi-2\lambda}\left(\frac{5}{\beta^2}-\frac{1}{\alpha^2}\right) - \frac{Q}{2}\frac{e^{-\Phi}}{\alpha^2\beta^2}\,,\\
\frac{df}{d\rho} &= \frac{1}{3}\frac{d\Phi}{d\rho} + \frac{2Q}{3}\frac{e^{-\Phi}}{\alpha^2\beta^2}\,.
\end{aligned}
\tag{17}
$$

In addition we obtain a differential equation for the ρ-dependence of the Killing spinor which can be easily integrated to yield

$$
\epsilon = e^{f/2}\epsilon_0\,,
\tag{18}
$$

where ϵ_0 is a constant spinor subject to the projections (16). In fact, this functional form of the Killing spinor can be deduced just from the supersymmetry algebra.

In order to solve the system (17) we found it useful to redefine our variables as

$$
\begin{aligned}
dr &= e^{-\Phi/3}d\rho\,, & A &= f-\Phi/3\,,\\
a_1 &= \alpha\, e^{-\Phi/3}\,, & a_2 &= \beta e^{-\Phi/3}\,,\\
a_3 &= e^{\lambda+2\Phi/3}\,, & a &= e^{-2\lambda+2\Phi/3}\,.
\end{aligned}
\tag{19}
$$

We also set for the rest of the paper the parameter $g=2$ since in any case in can be put to any value by appropriate rescalings. Using the results of [24], the eleven-dimensional metric takes the form

$$
ds^2_{11} = e^{2A}ds^2_{1,2} + dr^2 + a_1^2 d\Omega_2^2 + a_2^2 d\bar{\Omega}_2^2 + a_3^2 d\hat{\Omega}_2^2 + a^2\left(\hat{\sigma}_3-\sigma_3-\bar{\sigma}_3\right)^2,
\tag{20}
$$

where $\hat{\sigma}_i$ are left-invariant Maurer-Cartan SU(2) one-forms satisfying as a triplet the conditions $d\hat{\sigma}_i = \frac{1}{2}\epsilon_{ijk}\hat{\sigma}_j\wedge\hat{\sigma}_k$. We use the explicit representation

$$
\begin{aligned}
\hat{\sigma}_1 &= \cos\hat{\psi}\sin\hat{\theta}d\hat{\phi} - \sin\hat{\psi}d\hat{\theta}\\
\hat{\sigma}_2 &= \sin\hat{\psi}\sin\hat{\theta}d\hat{\phi} + \cos\hat{\psi}d\hat{\theta}\,,\\
\hat{\sigma}_3 &= d\hat{\psi} + \cos\hat{\theta}d\hat{\phi}\,,
\end{aligned}
\tag{21}
$$

so that the line element for the two-sphere $d\hat{\Omega}_2^2$ is given by an expression similar to the ones in (11). Using (8) we may compute the non-vanishing components of the four-form gauge field strength in eleven dimensions F_{0127}, where now all indices are in the tangent space. We easily find that

$$
F_{0127} = \frac{Q}{a_1^2a_2^2a_3^2a}\,.
\tag{22}
$$

Besides the metric and four-form, we may also use the fact that a Killing spinor can also be lifted from eight to eleven dimensions as $\epsilon_{11} = e^{-\Phi/6}\epsilon = e^{A/2}\epsilon_0$. Splitting the 32-component spinor ϵ_{11} as $\epsilon_{11} = \epsilon_{1,2} \times \epsilon_8$, one can show that the projections (16) leave two independent components in the 16-component spinor ϵ_8 whereas no restriction on the 2-component spinor $\epsilon_{1,2}$ is needed. Hence we are left with $\mathcal{N} = 2$ supersymmetry in three dimensions. We note that the proof of the above facts is completely parallel to the one we provided in our construction of G_2 holonomy manifolds from eight-dimensional gauged supergravity [19], and this is the reason why we do not repeat it here.

After the redefinitions (19), the system (17) becomes

$$
\begin{aligned}
a_1 \frac{da_1}{dr} &= \frac{a}{2} - \frac{Q}{3}\frac{1}{a_2^2 a_3^2 a}\,, \qquad \text{and cyclic in 1,2,3}\,, \\
\frac{da}{dr} &= 1 - \frac{a^2}{2}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2}\right) - \frac{Q}{3}\frac{1}{a_1^2 a_2^2 a_3^2}\,,
\end{aligned} \tag{23}
$$

whose solution determines the conformal factor as

$$
\frac{dA}{dr} = \frac{2Q}{3}\frac{1}{a_1^2 a_2^2 a_3^2 a}\,. \tag{24}
$$

1. The four-fold

Let us first concentrate to the case where the D2-branes are absent, i.e. when $Q = 0$. Then the general solution to the system (23) is

$$
\begin{aligned}
a_1^2 &= R^2 + l_1^2\,, \qquad & a_2^2 &= R^2 + l_2^2\,, \\
a_3^2 &= R^2\,, & a^2 &= R^2 U^2(R)\,,
\end{aligned} \tag{1.1}
$$

where

$$
U^2(R) = \frac{3R^4 + 4(l_1^2 + l_2^2)R^2 + 6 l_1^2 l_2^2 + 3C/R^4}{6(R^2 + l_1^2)(R^2 + l_2^2)}\,. \tag{1.2}
$$

We also observe the relation of the two variables r and R via the differentials

$$
dr = \frac{2}{\mathrm{U}(R)} dR\,. \tag{1.3}
$$

Here we have denoted three of the constants of integration by l_1, l_2 and C and we have absorbed the fourth one by an appropriate shift in the variable R.[2] We can also see that in this case $e^{2\Phi} = R^3\,\mathrm{U}(R)$, $f = \Phi/3$ and $A = 0$. Hence the lifted eleven dimensional solution in (20) factorizes into the three-dimensional flat space-time and a Calabi-Yau four-fold, recovering the lift in [22] from SU(3) holonomy in type-IIA string theory to SU(4) holonomy in M-theory. Topologically the Calabi-Yau four-fold is a $\mathbb{C}^2$ bundle over

[2] Note that in the metric (20) and in the system (23) the three two-spheres are completely equivalent with no distinction between a "space-time" and an internal two-sphere. This equivalence seems to be broken by the solution (1.1). However, this is only an artifact of setting the fourth integration constant to zero. The symmetry between the three two-spheres can be manifestly restored in the solution (1.1) if we make the variable shift $R^2 \to R^2 + l_3^2$ and simultaneously redefine $l_1^2 \to l_1^2 - l_3^2$ and $l_2^2 \to l_2^2 - l_3^2$.

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$S^2 \times S^2$ and was also constructed with a different method in [28] (for $C = 0$). This result includes those obtained in [13, 21], where both spheres in the four-cycle were taken to have the same radius, so that $l_1 = l_2$.

Let us note that for $R \to \infty$ the eight-dimensional metric takes the universal form

$$ds_8^2 \simeq R^2(d\Omega_2^2 + d\bar{\Omega}_2^2 + d\hat{\Omega}_2^2) + 8dR^2 + \frac{R^2}{2}(\hat{\sigma}_3 - \sigma_3 - \bar{\sigma}_3)^2\,, \qquad \text{as} \quad R \to \infty\,. \tag{1.4}$$

This solution is in fact exact for all values of R since it can be obtained from equations (1.1)-(1.2) by simply letting $l_1 = l_2 = C = 0$. However, extending it to the interior is problematic since we reach a singularity at $R = 0$, where the fiber, the S^2 and the four-cycle collapse to a point. Resolving the singularity to avoid this collapse requires that we turn on some of the different moduli parameters which also determine the behavior of the solution in the interior. In the following we further analyze the solution for different generic ranges of the parameters and variables in order to determine for which ones the singularity can indeed be resolved.

We may systematically discuss the different cases as follows:

- $l_1 = l_2 = 0$: In this case, when the constant $C \geq 0$, the variable $R \geq 0$ and then the manifold is singular at $R = 0$. If, however, $C = -a^8 < 0$, where a is a real positive constant, then the variable $R \geq a$. Changing to a new radial variable $t = 2\sqrt{a(R-a)}$ we find the behavior

  $$ds_8^2 \simeq a^2(d\Omega_2^2 + d\bar{\Omega}_2^2 + d\hat{\Omega}_2^2) + dt^2 + t^2(\hat{\sigma}_3 - \sigma_3 - \bar{\sigma}_3)^2\,, \qquad \text{as} \quad t \to 0\,. \tag{1.5}$$

  Therefore, near $t = 0$ (or equivalently $R = a$) and for constant θ and ϕ, as well as for the corresponding barred and hatted angles, the metric behaves as $dt^2 + t^2 d\hat{\psi}^2$ which shows that $t = 0$ is a *bolt* singularity [25] which is removable provided that the periodicity of the angle $\hat{\psi}$ is restricted to $0 \leq \hat{\psi} < 2\pi$. Then the space becomes topologically $S^2 \times S^2 \times S^2 \times \mathbb{R}^2$ and the full solution interpolates between this space for $R \to a$ and the four-fold (1.4) for $R \to \infty$.

- $l_1^2 > 0$ and $l_2^2 > 0$: In this case, when the constant $C > 0$, the variable $R \geq 0$ and there is a singularity at $R = 0$. If, however, $C = 0$ then we have the behavior

  $$ds_8^2 \simeq l_1^2 d\Omega_2^2 + l_2^2 d\bar{\Omega}_2^2 + 4dR^2 + R^2 d\hat{\Omega}_2^2 + R^2(\hat{\sigma}_3 - \sigma_3 - \bar{\sigma}_3)^2\,, \qquad \text{as} \quad R \to 0\,. \tag{1.6}$$

  Hence, for constant θ, ϕ and $\bar{\theta}, \bar{\phi}$ the metric behaves as $4dR^2 + R^2(\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2)$ which shows that we simply have a coordinate singularity in the polar coordinate system on an $\mathbb{R}^4$ centered at $R = 0$. This is the so called *nut* singularity [25], which is removable by adding the point $R = 0$ and changing to cartesian coordinates. Therefore near $R = 0$ the manifold becomes topologically $S^2 \times S^2 \times \mathbb{R}^4$. Then the full solution interpolates between this space for $R \to 0$ and the four-fold (1.4) for $R \to \infty$. If $C < 0$ then there is an R_0 such that $\mathrm{U}(R_0)^2 = 0$ (we take the largest root of this quartic, in R_0^2, equation) and therefore we have that $R \geq R_0$. Changing to a new radial variable $t = 2\sqrt{R_0(R - R_0)}$ we find the behavior

  $$ds_8^2 \simeq (R_0^2 + l_1^2)d\Omega_2^2 + (R_0^2 + l_2^2)d\bar{\Omega}_2^2 + R_0^2 d\hat{\Omega}_2^2 + dt^2 + t^2(\hat{\sigma}_3 - \sigma_3 - \bar{\sigma}_3)^2\,, \mathrm{as} t \to 0\,. \tag{1.7}$$

Hence the behavior is similar to that found before in (1.5), with a removable *bolt* singularity at $t = 0$. Hence it will not be discussed any further.

- $l_1^2 > 0$ and $l_2^2 < 0$ *or* $l_2^2 < l_1^2 < 0$: In this case it is convenient to define $\tilde{l}_2^2 = -l_2^2$ so that $\tilde{l}_2^2 > 0$. Then, when $C < \frac{1}{3}\tilde{l}_2^6(2l_1^2 + \tilde{l}_2^2)$, there is an $R_0 > \tilde{l}_2$ obtained as the largest root of the quartic (in R_0^2) equation $U^2(R_0) = 0$ and therefore the variable $R \geq R_0$. Then the behavior of the metric is given by (1.7) with the replacement $l_2^2 \to -\tilde{l}_2^2$ and as in that case, the *bolt* singularity at $R = R_0$ is removable. If $C = \frac{1}{3}\tilde{l}_2^6(2l_1^2 + \tilde{l}_2^2)$, then $R_0 \geq \tilde{l}_2$ and the metric near $R = R_0$ behaves as
$$ds_8^2 \simeq (R_0^2 - \tilde{l}_2^2)d\Omega_2^2 + R_0^2 d\hat{\Omega}_2^2 + dt^2 + \frac{t^2}{4}\left(\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + (\hat{\sigma}_3 - \sigma_3 - \bar{\sigma}_3)^2\right)\,, \qquad \text{as} \quad t \to 0\,, \tag{1.8}$$
where t is some new radial variable. Hence, for constant θ, ϕ and $\hat{\theta}, \hat{\phi}$ the metric behaves as $dt^2 + \frac{1}{4}t^2(\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \bar{\sigma}_3^2)$ which shows, as before in (1.6), that $t = 0$ is a removable *nut* singularity. Finally, if $C > \frac{1}{3}\tilde{l}_2^6(2l_1^2 + \tilde{l}_2^2)$, then $R \geq \tilde{l}_2$. We found that in this case we have a curvature singularity at $R = \tilde{l}_2$. We must note that in all of the above subcases the metric retains its euclidean signature even for $l_1^2 < 0$ as long as $l_1^2 + \tilde{l}_2^2 > 0$, which is in accordance with our original assumption.

2. Turning on the flux

Dealing with fluxes is always a much more involved problem. However we will provide now a quite promising method that can probably be extended in general to settings similar to the one studied in this note. Turning to the system (23) with $Q \neq 0$, we first perform the transformations
$$a_i = \tilde{a}_i e^{-Qx/3}\,, \qquad a = \tilde{a} e^{-Qx/3}\,, \tag{2.1}$$
where the new functions $\tilde{a}_i$, $i = 1, 2, 3$ and $\tilde{a}$ are to be determined and x is defined via the differential equation
$$\frac{dr}{dx} = a_1^2 a_2^2 a_3^2 a = \tilde{a}_1^2 \tilde{a}_2^2 \tilde{a}_3^2 \tilde{a} e^{-7Qx/3}\,. \tag{2.2}$$
Then we can deduce from (23) that the $\tilde{a}_i$'s and $\tilde{a}$ obey the system
$$\begin{aligned} \tilde{a}_i \frac{d\tilde{a}_i}{d\tilde{r}} &= \frac{\tilde{a}}{2}\,, \qquad i = 1, 2, 3\,, \\ \frac{d\tilde{a}}{d\tilde{r}} &= 1 - \frac{\tilde{a}^2}{2}\left(\frac{1}{\tilde{a}_1^2} + \frac{1}{\tilde{a}_2^2} + \frac{1}{\tilde{a}_3^2}\right)\,, \end{aligned} \tag{2.3}$$
where the new variable $\tilde{r}$ is related to r via $d\tilde{r} = dr e^{Qx/3}$. This system is the same as the one in (23), but with $Q = 0$. Hence we immediately conclude that the solution for the $\tilde{a}_i$'s and $\tilde{a}$ is given by (1.1)–(1.3) after the appropriate replacements of variables by the corresponding tilded ones. In addition, we deduce from (24) that
$$A = f - \frac{\Phi}{3} = \frac{2}{3}Qx\,. \tag{2.4}$$

It remains to relate the variables x and R, which is easily done using (2.2). The result is better expressed via the integral

$$e^{-2Qx} = -4Q \int \frac{dY}{Y^2\Big(Y^2 + \frac{4}{3}(l_1^2 + l_2^2)Y + 2l_1^2 l_2^2\Big) + C} + \text{const.}\,, \tag{2.5}$$

where $Y = R^2$. Evaluating this integral is elementary, but the general result is not very illuminating. Instead, we will consider some limiting cases. For $R \to \infty$, we have $e^{-2Qx} \simeq 4Q/3R^{-6} + \text{const.}$. Hence, x tends to a constant, which without loss of generality can be chosen to be zero. Therefore for $R \to \infty$ our solution becomes the Calabi-Yau four-fold times $\mathbb{E}_{2,1}$. Towards the infrared the details of the solution depend on the various integration parameters. We will consider first the case of $l_1 = l_2 = C = 0$. Then the solution we gave above for $R \to \infty$ is actually valid for all values of R. Then for $R \to 0$ (and assuming $Q > 0$) we have that $x \to -\infty$ and we obtain the eleven-dimensional direct product solution $AdS_4 \times Q^{1,1,1}$. The case that we have just described was considered in [21]. When $C = 0$, but for a generic choice of parameters l_1 and l_2 such that the radial variable $R \geq 0$, we have that $e^{-2Qx} \simeq \frac{2Q}{l_1^2 l_2^2 R^2}$, as $R \to 0$. Then for $R = 0$ there is a curvature singularity which however has a natural interpretation in terms of the D2-D6 system. In order to see that, consider the type IIB supergravity solution obtained from dimensional reducing (20) along the directions corresponding to the Killing vector field $\partial/\partial\hat{\psi}$. Then, for $R \to 0$ and after some algebraic manipulations, the metric becomes that for the D2-D6 system in the near horizon limit, with radial variable $r \sim R^2$ and harmonic functions $H_2, H_6 \sim 1/r$. It turns out that the constants l_1^2 and l_2^2 are naturally related to the ratio of D-brane charges.

3. Relation to Freund-Rubin compactifications

In order to study the stability properties of Freund-Rubin type compactifications [29] *finitely* away from the supersymmetric vacua, a number of four-dimensional supergravity actions have been constructed in the past by dimensionally reducing eleven-dimensional supergravity on seven-dimensional manifolds representing deformations of the well known supersymmetric vacua. In this way one obtains a four-dimensional theory of gravity coupled to scalars which model the deformations and which also self-interact via a potential. This program was initiated by Page [30] and further developed by Yasuda [31]. We think that it is important to make a precise connection with these works. We will restrict ourselves to the example considered in this paper, but other cases can be worked out in a similar fashion. In order to further dimensionally reduce eight-dimensional gauged supergravity on $S^2 \times S^2$, we start with the more general, than (10) and (12), ansatz

$$ds_8^2 = e^{-4h} G^{(4)}_{\mu\nu} dx^\mu dx^\nu + e^{2h+2\varphi} d\Omega_2^2 + e^{2h-2\varphi} d\bar{\Omega}_2^2\,, \qquad \mu,\nu = 0,1,2,7 \tag{3.1}$$

and

$$G_{x_0 x_1 x_2 x_7} = Q e^{-2\Phi - 12h} \sqrt{\det G^{(4)}}\,, \tag{3.2}$$

where the functions h, φ as well as the scalars Φ and λ depend only on the variables of the four-dimensional metric $G^{(4)}_{\mu\nu}$. and, as before, the four-form indices are curved. Then, dimensionally reducing (1), which is the relevant part of the eight-dimensional gauged supergravity action of [24], we obtain a four-dimensional action for gravity coupled to scalars of the form

$$S \sim \int d^4x \sqrt{\det G^{(4)}} \left(\frac{R^{(4)}}{4} + T - V \right) . \tag{3.3}$$

Specifically, the kinetic term for the scalars is

$$T = -3(\partial_\mu h)^2 - (\partial_\mu \varphi)^2 - \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{3}{2}(\partial_\mu \lambda)^2 , \tag{3.4}$$

whereas their potential reads

$$V = -e^{-6h} \cosh 2\varphi + \frac{1}{4} e^{2\Phi - 4\lambda - 8h} \cosh 4\varphi + \frac{1}{8} e^{-2\Phi - 4h}(e^{-8\lambda} - 4e^{-2\lambda}) + \frac{Q^2}{2} e^{-2\Phi - 12h} . \tag{3.5}$$

This four-dimensional theory should be the same as that constructed in [31] in relation to stability issues we mentioned of various compactifications of eleven-dimensional supergravity. In particular, eq.(19) of [31] (for $p = q = r = 1$) describes the four-dimensional theory obtained by a Freund-Rubin type compactification of eleven-dimensional supergravity on a deformed $Q^{1,1,1}$ space. In our normalization this action is of the form (3.3) with kinetic term

$$T = -\frac{63}{8}(\partial_\mu s)^2 - \frac{21}{8}(\partial_\mu u)^2 - \frac{3}{4}(\partial_\mu v)^2 - \frac{1}{4}(\partial_\mu w)^2 \tag{3.6}$$

and potential[3]

$$V = e^{9s} \left(-e^{u+v} \cosh w + \frac{1}{4} e^{8u+2v} \cosh 2w + \frac{1}{8} e^{8u-4v} - \frac{1}{2} e^{u-2v} \right) + \frac{Q^2}{2} e^{21s} . \tag{3.7}$$

We expect that the action (3.3) with kinetic and potential terms (3.4) and (3.5) will be identical to the action with corresponding terms given by (3.6) and (3.7). Indeed, we found that the appropriate field redefinitions are

$$w = 2\varphi , \qquad v = \frac{2}{3}(\lambda + \Phi - h) , \qquad u = \frac{2}{21}(2\Phi - 7\lambda - 2h) , \qquad s = -\frac{2}{7}\left(\frac{\Phi}{3} + 2h \right) . \tag{3.8}$$

Let us finally note that the potential (3.5) is derivable from the prepotential

$$W = -e^{-2h-\Phi} \left(e^{2\lambda} + \frac{1}{2} e^{-4\lambda} \right) - e^{\Phi - 2\lambda - 4h} \cosh 2\varphi + Q e^{-\Phi - 6h} , \tag{3.9}$$

using the appropriate for four space-time dimensions formula

$$V = \frac{1}{8} \left[\left(\frac{\partial W}{\partial \Phi} \right)^2 + \frac{1}{3} \left(\frac{\partial W}{\partial \lambda} \right)^2 + \frac{1}{6} \left(\frac{\partial W}{\partial h} \right)^2 + \frac{1}{2} \left(\frac{\partial W}{\partial \varphi} \right)^2 \right] - \frac{3}{8} W^2 . \tag{3.10}$$

The system of equations (17) can also be obtained from the four-dimensional action (3.3) as follows. First we make a domain wall ansatz for the metric

$$ds_4^2 = e^{2A_4} ds_{1,2}^2 + d\rho_4^2 , \tag{3.11}$$

[3]We differ from [31] by a factor of 2 which in not due to different normalization choices.

where the conformal factor A_4 depends only on the fourth radial coordinate ρ_4 and the same is true for the scalars. Then, first order equations for the scalars can be obtained using

$$\begin{aligned} \frac{d\Phi}{d\rho_4} &= \frac{1}{2}\frac{\partial W}{\partial \Phi}\,, \qquad & \frac{d\lambda}{d\rho_4} &= \frac{1}{6}\frac{\partial W}{\partial \lambda}\,, \\ \frac{dh}{d\rho_4} &= \frac{1}{12}\frac{\partial W}{\partial h}\,, \qquad & \frac{d\varphi}{d\rho_4} &= \frac{1}{4}\frac{\partial W}{\partial \varphi}\,, \end{aligned} \tag{3.12}$$

whereas the conformal factor is simply given by

$$\frac{dA_4}{d\rho_4} = -\frac{1}{2}W\,. \tag{3.13}$$

As usually, satisfying the above first order system implies that the second order equations of motion for the action (3.3) are automatically satisfied as well. We note at this point that we have constructed the prepotential W in order to reproduce the expression for the potential V using (3.10). This does not necessarily guarantee that the first order equations (3.12) imply supersymmetry. Nevertheless, this is true in our case. Indeed, taking into account the redefinitions $\alpha = e^{h+\varphi}$, $\beta = e^{h-\varphi}$, $f = A_4 - 2h$ and $d\rho = e^{-2h}d\rho_4$, we can verify that the system of (3.12) and (3.13) coincides with the system (17).

We note that the other cases that have been discussed in [21] concerning the squashed S^7 and $N^{0,1,0}$ in the infrared, can be obtained from four-dimensional supergravity actions using the method that we outlined here. In addition, these actions coincide with those obtained in the past [30, 31]. We also note that [31] contains a few other cases worth a reinterpretation in terms of branes with fluxes wrapped on supersymmetric cycles.

The eleven dimensional description of solutions to eight dimensional gauged supergravity corresponding to configurations of wrapped branes in the presence of fluxes provides a fruitful tool to approach compactifications of M-theory. In this note we have studied in detail the case of D6-branes wrapped on Kähler four-cycles, using as a representative example wrapping on $S^2 \times \bar{S}^2$, and proposed a clean method to construct solutions in the presence of four-form field strength. We hope similar studies can be performed for different configurations.

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